Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 58 (2008) 55–63
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Asymptotic properties of the massive scalar field in the external Schwarzschild spacetime

Hyungjin Huh

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

Received 31 July 2006; received in revised form 5 September 2007; accepted 18 September 2007
Available online 22 September 2007

Abstract

The asymptotic properties of the solution to the Klein–Gordon equation will be studied in the Schwarzschild spacetime background. The results are based on the global Sobolev-type inequalities and the generalized energy estimates.

MSC: 35B40; 35L15; 83C57

Keywords: Initial value problem; Massive scalar field; Black hole; Asymptotic behavior

1. Introduction

The goal of this paper is to derive asymptotic behaviors of the solution to the Klein–Gordon equation in the external Schwarzschild spacetime. The several decay estimates of the massive scalar field in Minkowski spacetime have been studied intensively [14,17].

The asymptotic behavior of the scalar field in the presence of a black hole has been studied by several authors [3, 4,8,12,13,24] who were mainly concerned with massless fields. However, considering the recent development of the Kaluza–Klein theories such as the Randall–Sundrum model [25] in string theory the evolution of massive scalar fields will become important. Moreover the physical mechanism of the massive scalar field may be qualitatively different from that of the massless one [18,20,21]. One of the features is that the decay in time of massive scalar field is much faster along null directions than along timelike directions. We also note that the slowest rate of decay occurs along null directions for the solution of the massless scalar field.

The Klein–Gordon equation in the Schwarzschild metrics has been studied by several authors [1,16,23]. The bound in the exterior Schwarzschild spacetime for the C^∞ solution of the covariant Klein–Gordon equation which has compact support on Cauchy surfaces in Kruskal spacetime was obtained in [16] and asymptotic behavior at the horizon and infinity was studied in [23].

On the other hand, D. Christodoulou and S. Klainerman derived in [5] the uniform asymptotic behavior of solutions to Maxwell equations and spin 2 equations in Minkowski space, based on generalized energy estimates and a systematic use of the invariance properties of the field equations with respect to the conformal group of the Minkowski

E-mail address: huh@math.kyoto-u.ac.jp.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.09.006

spacetime. Moreover they also showed in a remarkable paper [6] that the global nonlinear stability of Einstein equation around the Minkowski spacetime (see also [19]). The results were extended in [15] to the case of the Maxwell equations in the external Schwarzschild spacetime. We will adapt those methods to study the Klein–Gordon equation. However the application is not straightforward. The main difficulty comes from the fact that the energy–momentum tensor of the massive wave field is not traceless in contrast to the Maxwell field's, which is traceless. For the case of massless wave field, the modified vector field was suggested to make the error terms simple [5,22] and its idea was used in a black hole spacetime [7].

Recently the decay rates for massive Dirac particles in black hole geometry were shown in [10,11]. In fact it was proved that the Dirac wavefunction decays in $L^\infty_{\rm loc}$ at least at the rate $t^{-5/6}$ and the rate of decay is sharp for generic initial data. From the physical viewpoint, those results show that the black hole's gravitational attraction has an effect on the long-time behavior of the massive Dirac particles. Therefore we may expect that the local decay rate of the solution to (3) is at most $t^{-5/6}$ for generic initial data. Actually it was anticipated in [21] that the massive scalar field decays in $L^\infty_{\rm loc}$ at the same rate $t^{-5/6}$.

We also note interesting papers [2,9]. In particular it was shown in [2] that there are a countable family of globally regular solutions of spherically symmetric Einstein–Klein–Gordon equations. These solutions, known as mini-boson stars, are not static: although the metric and the stress-energy tensor of the scalar field are time independent, the scalar field itself has the form of a standing wave $\phi(r,t) = e^{i\omega t}\tilde{\phi}(r)$. This is in stark contrast with the classical case where Minkowski spacetime is the background metric.

In Section 2, our main result will be stated. In Section 3, we will describe several preliminary facts and show how to control the energy-type norms. In Section 4, the proof of Proposition 2.1 will be shown.

2. Statement of the result

To begin with we summarize several facts concerning the Klein–Gordon equation in the Schwarzschild metric. Recall that in Boyer–Lindquist coordinates (t, r, θ, ϕ) with $r > 0$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$, the Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where M denotes the mass of the black hole. The $r > 2M$ region, denoted by $\mathcal{M}$, is called the external Schwarzschild spacetime. We will use the notation $\Phi^2 = 1 - \frac{2M}{r}$. The Klein–Gordon equation in the Schwarzschild spacetime is

$$\Box_g u - m^2 u = g^{\alpha\beta} D_\alpha D_\beta u - m^2 u = 0, \quad (m \neq 0), \tag{2}$$

where D_α denotes the Levi-Civita connection relative to the Schwarzschild metric and $(g^{\alpha\beta})$ is an inverse of $(g_{\alpha\beta})$. The external Schwarzschild spacetime has the following geometric structures.

(a) Hyperplane: $\Sigma_t = \{p \in \mathcal{M} \mid t(p) = t\}$.

(b) Canonical null foliations consist of the null cones: $\{C(v), \underline{C}(\underline{v})\}$,

$$C(v) \equiv \{p \in \mathcal{M} \mid v(p) = v = t - r_*\},$$
$$\underline{C}(\underline{v}) \equiv \{p \in \mathcal{M} \mid \underline{v}(p) = \underline{v} = t + r_*\},$$

where $r_* \equiv r + 2M \log(\frac{r}{2M} - 1)$. We will use the following notation:

$$\Sigma_{t_0,\delta} = \{(t,x) | t = t_0, t - r_* \le -r_*(2M+\delta)\},$$
$$l^{t_1}_{t_0,\delta} = \{(t,x) | t_0 \le t \le t_1, t - r_* = -r_*(2M+\delta)\},$$
$$V^{t_1}_{t_0,\delta} = \{(t,x) | t_0 \le t \le t_1, t - r_* \le -r_*(2M+\delta)\},$$
$$R_\delta = \{(t,x) \in \mathcal{M} | t \ge 0 \text{ and } t + r_*(2M+\delta) - r_*(r) \le 0\}.$$

In particular, we define $r_0 = 2M + \delta_0$ by $r_*(2M + \delta_0) = 0$.

(c) Canonical sphere foliation: $S(t,r) = \Sigma_t \cap C(v)$ where $C(v)$ is the outgoing null cones, through (t, r). For each fixed t the family $\{S(t,r)\}$ produces a S^2 foliation of the hyperplane Σ_t.

(d) Canonical null pairs are given by the vector fields

$$e_3 = \Phi^{-1}\partial_t - \Phi\partial_r, \qquad e_4 = \Phi^{-1}\partial_t + \Phi\partial_r.$$

We can complete the pair e_3, e_4 to a null frame $\{e_1, e_2, e_3, e_4\}$, at a generic point p, by taking an orthonormal frame $\{e_a\}$, $a \in (1, 2)$[1] on the space tangent to the sphere $S(t, r)$ passing through p.

(e) Conformal structure: the external Schwarzschild spacetime has the following Killing vector fields.

1. Time translation

$$T = \partial_t.$$

2. The three generators of the rotation group O $(=\{O_{ij} : 1 \leq i < j \leq 3\})$,

$$O_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad \text{for } i, j = 1, 2, 3.$$

For convenience, we will use the following notation:

$$\tilde{T} = \Phi^{-1} T, \qquad \tilde{N} = \Phi N = \Phi\partial_r.$$

Now let us state our result.

Proposition 2.1. *Consider the following initial value problem:*

$$\begin{cases} g^{\alpha\beta} D_\alpha D_\beta u - m^2 u = 0, \quad (m > 0), \\ u(0, x) = u_0(x), \quad D_t u(0, x) = u_1(x), \end{cases} \tag{3}$$

in the external Schwarzschild spacetime. Let δ be a positive number. If a point (t, r) belongs to a region R_δ and u is a solution of the Cauchy problem equation (3), then the following estimates hold:

(I) *For a fixed δ $(\geq\delta_0)$, we have*

$$\sup_{S(t,r)} (r\tau^{\frac{1}{2}}|u|) \leq c\sqrt{\frac{m^2+1}{m^2}} I^{\frac{1}{2}}, \tag{4}$$

where $\tau^2 = 1 + (t - r_*)^2$ and c is independent of δ $(\geq\delta_0)$. The quantity I is defined as (13).

(II) *For a fixed δ $(\leq\delta_0)$, we have*

$$\sup_{S(t,r)} (r\tau^{\frac{1}{2}}|u|) \leq cA(m, \delta)^{\frac{1}{2}} I^{\frac{1}{2}}, \tag{5}$$

where $A(m, \delta) = \frac{m^2+1}{m^2} (\frac{2M+\delta}{\delta})^{\frac{3}{2}} (1 - r_*(2M + \delta))^2$.

Remark. The estimate (II) is actually concerned with the region $R_\delta - R_{\delta_0}$ $(\delta \leq \delta_0)$ and note that $A(m, \delta) \to \infty$ as $\delta \to 0$.

3. Preliminaries

We recall several facts and estimates used for the proof of Proposition 2.1. The following lemma states the commutativity of the Klein–Gordon operator and the Lie derivatives with respect to T, O_{ij}.

Lemma 3.1. *If u is a solution to the Klein–Gordon equation, then Lie derivatives of u with respect to T, O_{ij} are also solutions.*

[1] For instance, $e_\theta = \frac{1}{r}\partial_\theta$, $e_\phi = \frac{1}{r\sin\theta}\partial_\phi$.

Proof. Recall that

$$\Box_g u = g^{\alpha\beta} D_\alpha D_\beta u = g^{\alpha\beta}(\partial_{\alpha\beta} u - \Gamma^\gamma_{\alpha\beta}\partial_\gamma u) = \Box' u - g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma u,$$

where Γ denotes the Christoffel symbol with regard to the given metric. Then we have

$$\begin{aligned}\partial_t(\Box_g u - m^2 u) &= \partial_t(\Box' u) - \partial_t(g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma u) - m^2(\partial_t u)\\ &= \Box'(\partial_t u) - g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma(\partial_t u) - m^2(\partial_t u)\\ &= \Box_g(\partial_t u) - m^2(\partial_t u),\end{aligned}$$

where the second equality above comes from the fact that $g^{\alpha\beta}$ and $\Gamma^\gamma_{\alpha\beta}$ are independent of a time variable t. In fact, we can check that the only nonzero components of Christoffel symbol are

$$\begin{aligned}&\Gamma^1_{00} = \Phi^2 M/r^2, \qquad \Gamma^1_{11} = -\Phi^{-2} M/r^2,\\ &\Gamma^1_{22} = -r\Phi^2, \qquad \Gamma^1_{33} = -\Phi^2 r\sin^2\theta, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta.\end{aligned}$$

Next we will check the case $[O_{12}, \Box_g - m^2] = 0$. We know that $O_{12} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} = \partial_\phi$.

$$\begin{aligned}O_{12}(\Box_g u - m^2 u) &= O_{12}(\Box' u) - O_{12}(g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma u) - m^2 O_{12} u\\ &= \Box'(O_{12} u) - g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma(O_{12} u) - m^2 O_{12} u\\ &= \Box_g(O_{12} u) - m^2(O_{12} u).\end{aligned}$$

By the similar argument we can prove the other case O_{23}, O_{13}. □

To define conserved positive quantities, the energy–momentum tensor will be made use of.

Definition 3.2. The energy–momentum tensor of a function u is

$$Q_{\mu\nu}[u] = D_\mu u D_\nu u - \frac{1}{2} g_{\mu\nu}(D^\alpha u D_\alpha u + m^2 u^2).$$

Lemma 3.3. 1. *The energy–momentum tensor Q is symmetric and satisfies $Q(X, Y) \geq 0$ for any non-spacelike future directed vector fields X, Y.*

2. *If u is a solution of the Klein–Gordon equation then the energy–momentum tensor Q has vanishing divergence.*

Proof. 1. The future directed vector fields can be written as linear combinations of e_3, e_4 with nonnegative coefficients. So the result is an immediate consequence of the following:

$$Q_{33} = Q_{\mu\nu} e_3^\mu e_3^\nu = |e_3(u)|^2, \tag{6}$$

$$Q_{44} = Q_{\mu\nu} e_4^\mu e_4^\nu = |e_4(u)|^2, \tag{7}$$

$$Q_{34} = Q_{\mu\nu} e_3^\mu e_4^\nu = |\nabla\!\!\!/ u|^2 + m^2 u^2, \tag{8}$$

where $\nabla\!\!\!/$ is the covariant derivative of the two-dimensional submanifold $S(t, r)$ with respect to the induced metric.

2. The second result can be shown as follows:

$$\begin{aligned}D^\alpha Q_{\alpha\beta} &= D^\alpha D_\alpha u D_\beta u + D_\alpha u D^\alpha D_\beta u - \frac{1}{2}\left(D_\beta D^\mu u D_\mu u + D^\mu u D_\beta D_\mu u\right) - m^2 u D_\beta u\\ &= D_\alpha u D^\alpha D_\beta u - D^\alpha u D_\beta D_\alpha u\\ &= D_\alpha u(D^\alpha D_\beta u - D_\beta D^\alpha u) = 0. \quad \Box\end{aligned}$$

Remark. Q is not traceless. In fact, $Q_{\alpha\beta} g^{\alpha\beta} = -D^\mu u D_\mu u - 2m^2 u^2$.

Now we introduce a global Sobolev-type inequality. Using $\tilde{T}$ we can define positive definite metric $\overline{g}$ in $\mathcal{M}$:

$$\overline{g}_{\mu\nu} = g_{\mu\nu} + 2\tilde{T}_\mu \tilde{T}_\nu.$$

If U is an arbitrary tensor field in $\mathcal{M}$ of type (q, p) we shall denote by $|U|$ its pointwise norm relative to $\overline{g}$:

$$|U| = \left(\overline{g}^{\mu_1\nu_1}\cdots\overline{g}^{\mu_p\nu_p}\overline{g}_{\alpha_1\beta_1}\cdots\overline{g}_{\alpha_q\beta_q}U^{\alpha_1\cdots\alpha_q}_{\mu_1\cdots\mu_p}U^{\beta_1\cdots\beta_q}_{\nu_1\cdots\nu_p}\right)^{1/2}.$$

We denote by $\not{D}_{\tilde{N}}$ the projection over the tangent space of $S(t, r)$ of $D_{\tilde{N}}$ and by $\not\nabla$ the Levi-Civita connection relative to the induced metric on $S(t, r)$. To get an L^∞ estimate of the solution to the Klein–Gordon equation, we shall use the global Sobolev-type inequality which was found in [6,15].

Lemma 3.4. *Let U be a C^∞ tensor field tangential to $S(t, r)$ and satisfying*

$$\lim_{r\to\infty} r|U(r,\omega)| = 0,$$

where ω indicates the angular coordinates; then the following inequality holds:

$$\sup_{S(t,r)}(r\tau^{\frac{1}{2}}|U|) \le c\left[\int_{\Sigma_t([r,\infty))}|U|^2 + r^2|\not\nabla U|^2 + \tau^2|\not{D}_{\tilde{N}}\,U|^2 + r^4|\not\nabla^2 U|^2 + r^2\tau^2|\not\nabla\not{D}_{\tilde{N}}\,U|^2\right]^{\frac{1}{2}},$$

where $\tau^2 = 1 + (t - r_*)^2$ and the integral over the spacelike hypersurface Σ_t is defined as

$$\int_{\Sigma_t([r,\infty))} H \equiv \int_r^\infty \mathrm{d}r' \int_{S(t,r')} \Phi^{-1} H.$$

Now we introduce how to control the energy-type norms. Recall that the energy–momentum tensor of a function u relevant to the Klein–Gordon equation is given by

$$Q_{\mu\nu}[u] = D_\mu u D_\nu u - \frac{1}{2}g_{\mu\nu}(D^\alpha u D_\alpha u + m^2u^2).$$

Define the covariant vector field P associated with vector field X, $P_\mu[u] = Q_{\mu\nu}[u]X^\nu$; then we can get the following equality easily:

$$D^\alpha P_\alpha = D^\alpha Q_{\alpha\beta}X^\beta + \frac{1}{2}Q^{(X)}_{\alpha\beta}\pi^{\alpha\beta},$$

where ${}^{(X)}\pi^{\alpha\beta}$ is a deformation tensor $\mathcal{L}_X g$. Integrating $D^\alpha P_\alpha$ over $V^{t_1}_{t_0,\delta}$ we obtain the following lemma.

Lemma 3.5. *Let $P_\mu[u] = Q_{\mu\nu}[u]X^\nu$; then Stokes's theorem implies*

$$\int_{\Sigma_{t_1,\delta}} Q(X,\tilde{T}) + \int_{l^{t_1}_{t_0,\delta}} \Phi^{-1}Q(X,e_4) = \int_{\Sigma_{t_0,\delta}} Q(X,\tilde{T}) - \int_{V^{t_1}_{t_0,\delta}} D^\alpha P_\alpha. \tag{9}$$

We will assign special timelike vector fields to X. The elementary application is $X = T$. Since T is a Killing vector field, if u is a solution of the Klein–Gordon equation then Lemma 3.3 guarantees that $D^\alpha P_\alpha$ vanishes. Moreover the positivity property of the energy–momentum tensor Q relative to future causal vector fields T, e_4 makes the second integral of equation (9) nonnegative. So we obtain the following inequality:

$$\int_{\Sigma_{t,\delta}} Q(T,\tilde{T}) \le \int_{\Sigma_{t_0,\delta}} Q(T,\tilde{T}). \tag{10}$$

However estimate (10) is not enough to control the weighted integral in Lemma 3.4. To obtain information concerning the behavior of u along null directions we borrow some ideas from [18]. That is, we consider the case of $X = f\cdot T$ where f depends only on r, t; then we can calculate $D^\alpha P_\alpha$ as follows:

$$D^\alpha P_\alpha = -\frac{\Phi}{4}\left(Q_{33}e_4(f) + Q_{34}e_4(f) + Q_{43}e_3(f) + Q_{44}e_3(f)\right). \tag{11}$$

Then we obtain the following inequality:

$$\int_{\Sigma_{t,\delta}} Q(fT, \tilde{T}) \leq \int_{\Sigma_{t_0,\delta}} Q(fT, \tilde{T}) - \int_{V_{t_0,t_1,\delta}} D^\alpha P_\alpha. \tag{12}$$

Let us consider the following integrals:

$$\int_{\Sigma_t} Q[L_T^a L_O^b u](X, \tilde{T}) \quad \text{for } a, b : \text{ nonnegative integer},$$

where L_V is the Lie derivative with respect to a vector field V. From the integrals we define the following norm:

$$I(\Sigma_t, X) = \sum_{0 \leq a+b \leq 2} \int_{\Sigma_t} Q[L_T^a L_O^b u](X, \tilde{T}). \tag{13}$$

We consider the Cauchy problem for the Klein–Gordon equation in the Schwarzschild spacetime where the initial data are given on the Σ_0 hypersurface. We assign initial data for which the value $I := I(\Sigma_0, X)$ with $X = (1 + (t - r_*)^2)T = \tau^2 T$ is finite.

4. Proof of Proposition 2.1

Now we prove Proposition 2.1. Recall the global Sobolev inequality

$$\sup_{S(t,r)} (r\tau^{1/2}|u|) \leq c \left[\int_{\Sigma_t[r,\infty)} |u|^2 + r^2 |\nabla\!\!\!/ u|^2 + \tau^2 |D\!\!\!/_{\tilde{N}}\, u|^2 + r^4 |\nabla\!\!\!/^2\, u|^2 + r^2 \tau^2 |\nabla\!\!\!/ D\!\!\!/_{\tilde{N}}\, u|^2 \right]^{\frac{1}{2}}. \tag{14}$$

If we can control the right hand side of (14) in terms of the initial data then the proof is done. Moreover let us recall that a point (t, r) belongs to a region R_δ.

For the first integral in the right hand side of (14), using (10) the following estimate can be obtained easily (note that $Q(T, \tilde{T}) = \frac{\Phi}{4}(Q_{33} + Q_{44} + 2Q_{34})$):

$$\begin{aligned} \int_{\Sigma_t[r,\infty)} |u|^2 &\leq \frac{1}{m^2} \int_{\Sigma_t[r,\infty)} Q_{34}[u] \leq \frac{2}{m^2} \int_{\Sigma_t[r,\infty)} \Phi^{-1} Q[u](T, \tilde{T}) \\ &\leq \frac{2}{m^2} \Phi^{-1}(\delta) \int_{\Sigma_t[r,\infty)} Q[u](T, \tilde{T}) \\ &\leq \frac{2}{m^2} \Phi^{-1}(\delta) \int_{\Sigma_0} Q[u](T, \tilde{T}) \leq \frac{2}{m^2} \Phi^{-1}(\delta) I, \end{aligned} \tag{15}$$

where we use the notation $\Phi^{-1}(\delta) = \sqrt{\frac{2M+\delta}{\delta}}$ $(= \Phi^{-1}(2M + \delta))$ for brevity.

For the second integral of (14), we can check by direct calculations that

$$|L_O u|^2 = r^2 |\nabla\!\!\!/ u|^2 \quad \text{for any function } u, \tag{16}$$

where the covariant derivative $\nabla\!\!\!/$ is calculated with respect to the metric (1) (see also [15]). Therefore we have

$$\begin{aligned} \int_{\Sigma_t([r,\infty))} r^2 |\nabla\!\!\!/ u|^2 &\leq \frac{1}{m^2} \int_{\Sigma_t([r,\infty))} Q[L_O u](e_3, e_4) \leq \frac{2}{m^2} \int_{\Sigma_t([r,\infty))} \Phi^{-1} Q[L_O u](T, \tilde{T}) \\ &\leq \frac{2}{m^2} \Phi^{-1}(\delta) \int_{\Sigma_0} Q[L_O u](T, \tilde{T}) \leq \frac{2}{m^2} \Phi^{-1}(\delta) I. \end{aligned}$$

Using the following identities together with (16):

$$|\nabla\!\!\!/ L_O u|^2 = |L_O \nabla\!\!\!/ u|^2 \quad \text{for any function } u, \tag{17}$$

$$r^2 |\nabla\!\!\!/ U|^2 = |L_O U|^2 + |U|^2 \quad \text{for any 1-form } U, \tag{18}$$

we may control the fourth integral of (14) in a similar way.

$$\int_{\Sigma_t([r,\infty))} r^4|\nabla\!\!\!/^2 u|^2 \le \int_{\Sigma_t([r,\infty))} r^2|L_O \nabla\!\!\!/ u|^2 + r^2|\nabla\!\!\!/ u|^2 \le \int_{\Sigma_t([r,\infty))} |L_O^2 u|^2 + |L_O u|^2$$
$$\le \frac{1}{m^2}\int_{\Sigma_t([r,\infty))} Q[L_O^2 u](e_3,e_4) + Q[L_O u](e_3,e_4) \le \frac{2}{m^2}\Phi^{-1}(\delta) I.$$

The third and fifth integrals in (14) are different from the previous ones because $D^\alpha P_\alpha$ does not vanish. Let us show this with the third integral first. We know that

$$|D\!\!\!/_{\tilde N}\, u|^2 \le 2\left(|\Phi^{-1} L_T u|^2 + |e_4(u)|^2\right) \quad \text{for any function } u. \tag{19}$$

Therefore,

$$\int_{\Sigma_t([r,\infty))} \tau^2 |D\!\!\!/_{\tilde N}\, u|^2 \le 2\int_{\Sigma_t([r,\infty))} \tau^2 \left(\Phi^{-2}\frac{1}{m^2} Q[L_T u](e_3,e_4) + Q[u](e_4,e_4)\right)$$
$$\le 2\int_{\Sigma_t([r,\infty))} \tau^2 \left(2\Phi^{-3}\frac{1}{m^2} Q[L_T u](T,\tilde T) + 4\Phi^{-1} Q[u](T,\tilde T)\right).$$

We will show how to control $\int_{\Sigma_t([r,\infty))} \tau^2 Q[u](T,\tilde T)$. By Lemma 3.5 we obtain

$$\int_{\Sigma_t([r,\infty))} \tau^2 Q[u](T,\tilde T) \le \int_{\Sigma_0} \tau^2 Q[u](T,\tilde T) - \int_{V^t_{t_0,\delta}} \mu, \tag{20}$$

where $\mu = D^\alpha P_\alpha[u] = (r_* - t)(Q_{34}[u] + Q_{44}[u])$. Here is the place where we need to consider the region R_δ.

(A) The case $\delta_0 \le \delta$ will be considered for the time being. Since $S(t,r)$ belongs to R_{δ_0}, $r_* - t$ is positive. So the left hand side of (20) is controlled trivially by the initial data. The other term $\int_{\Sigma_t([r,\infty))} \tau^2 Q[L_T u](T,\tilde T)$ can be estimated in a similar process. Therefore, we obtain the following consequence:

$$\int_{\Sigma_t([r,\infty))} \tau^2 |D\!\!\!/_{\tilde N}\, u|^2 \le 8\Phi^{-1}(\delta)\int_{\Sigma_0} \tau^2 Q[u](T,\tilde T) + \frac{4}{m^2}\Phi^{-3}(\delta)\int_{\Sigma_0} \tau^2 Q[L_T u](T,\tilde T)$$
$$\le 8\left(\Phi^{-1}(\delta) + \frac{1}{m^2}\Phi^{-3}(\delta)\right) I \le c\frac{m^2+1}{m^2} I.$$

The identity that we need to consider to control the fifth integral in (14) is the commutative property of L_O and $D\!\!\!/_{\tilde N}$:

$$[L_O, D\!\!\!/_{\tilde N}] = 0 \quad \text{for any function } u. \tag{21}$$

Using (18), (19) and (21) we deduce the inequality

$$\tau^2 r^2 |\nabla\!\!\!/ D\!\!\!/_{\tilde N}\, u|^2 = \tau^2 |L_O D\!\!\!/_{\tilde N}\, u|^2 + \tau^2 |D\!\!\!/_{\tilde N}\, u|^2$$
$$\le 2\tau^2\left(|\Phi^{-1} L_T L_O u|^2 + |e_4(L_O u)|^2\right) + 2\tau^2\left(|\Phi^{-1} L_T u|^2 + |e_4(u)|^2\right).$$

We just show how to control $\tau^2 |L_O D\!\!\!/_{\tilde N}\, u|^2$ and the term $\tau^2 |D\!\!\!/_{\tilde N}\, u|^2$ can be estimated in a similar way.

$$\int_{\Sigma_t([r,\infty))} \tau^2 |L_O D\!\!\!/_{\tilde N}\, u|^2 \le 2\int_{\Sigma_t([r,\infty))} \tau^2\left(\Phi^{-2}|L_T L_O u|^2 + |e_4(L_O u)|^2\right)$$
$$\le 2\int_{\Sigma_t([r,\infty))} \tau^2\left(\Phi^{-2}\frac{1}{m^2} Q[L_T L_O u](e_3,e_4) + Q[L_O u](e_4,e_4)\right)$$
$$\le 2\int_{\Sigma_t([r,\infty))} \tau^2\left(\frac{2}{m^2}\Phi^{-3} Q[L_T L_O u](T,\tilde T) + 4\Phi^{-1} Q[L_O u](T,\tilde T)\right)$$
$$\le \frac{4}{m^2}\Phi^{-3}(\delta)\int_{\Sigma_0} \tau^2 Q[L_T L_O u](T,\tilde T) + 8\Phi^{-1}(\delta)\int_{\Sigma_0} \tau^2 Q[L_O u](T,\tilde T)$$
$$\le c\frac{m^2+1}{m^2} I,$$

where the assumption $\delta_0 \leq \delta$ is used in the last inequality. Therefore we reach the first result (I) in Proposition 2.1. Next we derive the second estimate (II) in Proposition 2.1 to which the region $\delta \leq \delta_0$ is related.

(B) The only troublesome items are

$$\int_{\Sigma_t([r,\infty))} \tau^2 |\not{D}_{\tilde{N}}\, u|^2 \quad \text{and} \quad \int_{\Sigma_t([r,\infty))} r^2\tau^2 |\not{\nabla}\not{D}_{\tilde{N}}\, u|^2.$$

As in the previous case, we will control the following integrals to estimate $\int_{\Sigma_t([r,\infty))} \tau^2 |\not{D}_{\tilde{N}}\, u|^2$:

$$\int_{\Sigma_t([r,\infty))} \Phi^{-3}\tau^2 Q[L_T u](T,\tilde{T}) \quad \text{and} \quad \int_{\Sigma_t([r,\infty))} \Phi^{-1}\tau^2 Q[u](T,\tilde{T}).$$

Using the notation $r_{t,0}$ for the radial coordinate of $S(t,0) = \Sigma_t \cap C(0)$, we know that

$$\begin{aligned}\int_{\Sigma_t([r,\infty))} \tau^2 Q[u](T,\tilde{T}) &= \int_{\Sigma_t([r,r_{t,0}])} \tau^2 Q[u](T,\tilde{T}) + \int_{\Sigma_t(r_{t,0},\infty)} \tau^2 Q[u](T,\tilde{T})\\ &\leq \tau^2(\delta)\int_{\Sigma_t([r,r_{t,0}])} Q[u](T,\tilde{T}) + \int_{\Sigma_t(r_{t,0},\infty)} \tau^2 Q[u](T,\tilde{T})\\ &\leq \tau^2(\delta)\int_{\Sigma_t([r,\infty))} Q[u](T,\tilde{T}) + \int_{\Sigma_0} \tau^2 Q[u](T,\tilde{T})\\ &\leq \tau^2(\delta)\int_{\Sigma_0} Q[u](T,\tilde{T}) + \int_{\Sigma_0} \tau^2 Q[u](T,\tilde{T})\\ &\leq cA(m,\delta)I,\end{aligned}$$

where $\tau^2(\delta) = 1 + r_*^2(2M+\delta)$ and $A(m,\delta) = \frac{m^2+1}{m^2}(\frac{2M+\delta}{\delta})^{\frac{3}{2}}(1 - r_*(2M+\delta))^2$. The same argument can be easily applied to the other integral $\int_{\Sigma_t([r,\infty))} \tau^2 Q[L_T u](T,\tilde{T})$. Considering the identities (18), (19) and (21) the following estimate of $\int_{\Sigma_t([r,\infty))} r^2\tau_-^2 |\not{\nabla}\not{D}_{\tilde{N}}\, u|^2$ can be easily obtained:

$$\begin{aligned}\int_{\Sigma_t([r,\infty))} r^2\tau_-^2 |\not{\nabla}\not{D}_{\tilde{N}}\, u|^2 &\leq 2\int_{\Sigma_t([r,\infty))} \tau^2\left(|\Phi^{-1}L_T L_O u|^2 + |e_4(L_O u)|^2 + |\Phi^{-1}L_T u|^2 + |e_4(u)|^2\right)\\ &\leq 2\int_{\Sigma_t([r,r_{t,0}])} \tau^2\left(|\Phi^{-1}L_T L_O u|^2 + |e_4(L_O u)|^2 + |\Phi^{-1}L_T u|^2 + |e_4(u)|^2\right)\\ &\quad + 2\int_{\Sigma_t(r_{t,0},\infty)} \tau^2\left(|\Phi^{-1}L_T L_O u|^2 + |e_4(L_O u)|^2 + |\Phi^{-1}L_T u|^2 + |e_4(u)|^2\right)\\ &\leq cA(m,\delta)I.\end{aligned}$$

Acknowledgement

The author is a JSPS Postdoctoral Fellow supported by a Grant-in-Aid.

References

[1] A. Bachelot, Asymptotic completeness for the Klein–Gordon equation on the Schwarzschild metric, Ann. Inst. H. Poincaré Phys. Théor. 61 (1994) 411–441.
[2] P. Bizoń, A. Wasserman, On existence of mini-boson stars, Comm. Math. Phys. 215 (2000) 357–373.
[3] P. Blue, A. Soffer, Semilinear wave equations on the Schwarzschild manifold: I. Local decay estimates, Adv. Differential Equations 8 (2003) 595–614.
[4] L.M. Burko, A. Ori, Late-time evolution of nonlinear gravitational collapse, Phys. Rev. D 56 (1997) 7820–7832.
[5] D. Christodoulou, S. Klainerman, Asymptotic properties of linear field equations in Minkowski space, Comm. Pure Appl. Math. 43 (1990) 137–199.
[6] D. Christodoulou, S. Klainerman, The global nonlinear stability of the Minkowski space, in: Princeton Mathematical Series, vol. 41, 1993.
[7] M. Dafermos, I. Rodnianski, The red-shift effect and radiation decay on black hole spacetimes. arXiv. gr-qc/0512119.
[8] J. Dimock, Scattering for the wave equation on the Schwarzschild metric, Gen. Relativity Gravitation 17 (1985) 353–369.

[9] F. Finster, J. Smoller, S.-T. Yau, Non-existence of time-periodic solutions of the Dirac equation in a Reissner–Nordstrom black hole background, J. Math. Phys. 41 (4) (2000) 2173–2194.
[10] F. Finster, N. Kamran, J. Smoller, S.-T. Yau, Decay rates and probability estimates for massive Dirac particles in the Kerr–Newman black hole geometry, Comm. Math. Phys. 230 (2002) 201–244.
[11] F. Finster, N. Kamran, J. Smoller, S.-T. Yau, The long-time dynamics of Dirac particles in the Kerr–Newman black hole geometry, Adv. Theoret. Math. Phys. 7 (1) (2003) 25–52.
[12] F. Finster, N. Kamran, J. Smoller, S.-T. Yau, An integral spectral representation of the propagator for the wave equation in the Kerr geometry, Comm. Math. Phys. 260 (2) (2005) 257–298.
[13] C. Gundlach, R.H. Price, J. Pullin, Large-time behavior of stellar collapse and expansion. I. Linearized perturbations, Phys. Rev. D 49 (1994) 883–889.
[14] L. Hörmander, Lectures on Nonlinear Hyperbolic Differential Equations, Springer-Verlag, Berlin, 1997.
[15] W. Inglese, F. Nicolò, Asymptotic properties of the electromagnetic field in the external Schwarzschild spacetime, Ann. H. Poincaré 1 (2000) 895–944.
[16] B.S. Kay, R.M. Wald, Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere, Classical Quantum Gravity 4 (1987) 893–898.
[17] S. Klainerman, Uniform decay estimates and the Lorentz invariance of the classical wave equation, Comm. Pure Appl. Math. 38 (1985) 321–332.
[18] S. Klainerman, Remark on the asymptotic behavior of the Klein Gordon equation in $\mathbb{R}^{n+1}$, Comm. Pure Appl. Math. 46 (1993) 137–144.
[19] S. Klainerman, F. Nicolò, On local and global aspects of the Cauchy problem in general relativity, Classical Quantum Gravity 16 (1999) R73–R157.
[20] H. Koyama, A. Tomimatsu, Asymptotic power-law tails of massive scalar fields in a Reissner–Nordström background, Phys. Rev. D 63 (2001) 064032.
[21] H. Koyama, A. Tomimatsu, Asymptotic tails of massive scalar fields in a Schwarzschild background, Phys. Rev. D 64 (2001) 044014.
[22] C. Morawetz, The limiting amplitude principle, Comm. Pure Appl. Math. 15 (1962) 349–362.
[23] J.-P. Nicolas, Non linear Klein–Gordon equation on Schwarzschild-like metrics, J. Math. Pures Appl. 74 (1995) 35–58.
[24] R.H. Price, Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations, Phys. Rev. D 5 (1972) 2419–2438.
[25] L. Randall, R. Sundrum, An alternative to compactification, Phys. Rev. Lett. 83 (1999) 4690–4693.